

Integrable Systems and Metrics of Constant Curvature

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Abstract.

In this article we present a Lagrangian representation for evolutionary systems with a Hamiltonian structure determined by a differential-geometric Poisson bracket of the first order associated with metrics of constant curvature. Kaup-Boussinesq system has three local Hamiltonian structures and one nonlocal Hamiltonian structure associated with metric of constant curvature. Darboux theorem (reducing Hamiltonian structures to canonical form "d/dx" by differential substitutions and reciprocal transformations) for these Hamiltonian structures is proved.

1 Introduction

In this article we describe nonlocal Hamiltonian structure associated with differential-geometric Poisson bracket of the first order with metric of constant curvature and its Lagrangian representation for evolutionary systems

$$u_t^k = f^k(\mathbf{u}, \mathbf{u}_x, \dots). \quad (1.1)$$

It means that (1.1) can be re-written as

$$u_t^i = \{u^i, H\} = \hat{A}^{ik} \frac{\delta H}{\delta u^k}, \quad (1.2)$$

where \hat{A}^{ik} is a Hamiltonian operator, $H = \int h(\mathbf{u}, \mathbf{u}_x, \dots) dx$ is a functional of conservation law density $h(\mathbf{u}, \mathbf{u}_x, \dots)$ and $\{u^i(x), u^j(x')\}$ is a Poisson bracket. Then we can introduce new variables $a^\alpha(\mathbf{u}) = \partial_x \varphi^\alpha$, where the system (1.1) is determined by action

$$S = \int L(\varphi_t, \varphi_x, \varphi_{xx}, \varphi_{xxx}, \dots) dx dt, \quad (1.3)$$

where $L(\varphi_t, \varphi_x, \varphi_{xx}, \varphi_{xxx}, \dots)$ is a Lagrangian.

The modern theory of Hamiltonian and Symplectic structures, Poisson brackets and Lagrangian representations is developed in [1] by L.D.Faddeev and V.E.Zakharov in 1971, where they showed that the Korteweg-de Vries equation has Poisson bracket

$$\{a(x), a(x')\} = \partial_x \delta(x - x') \quad (1.4)$$

determining the first Hamiltonian structure

$$a_t = \partial_x \frac{\delta H}{\delta a}, \quad (1.5)$$

where (in general case) the Hamiltonian is $H = \int h(\mathbf{a}, \mathbf{a}_x, \dots) dx$. N -component generalization of this formula on arbitrary dependence $u = u(\mathbf{a}(x))$ was established in the article [2] by B.A.Dubrovin and S.P.Novikov in 1983

$$\{u^i(x), u^j(x')\} = [g^{ij}(\mathbf{u}(x))\partial_x - g^{is}\Gamma_{sk}^j u_x^k] \delta(x - x'), \quad (1.6)$$

where $g^{ij}(\mathbf{u})$ is nondegenerated symmetric flat metric, Γ_{jk}^i are the coefficients of the corresponding Levi-Civita connection, $\Gamma_{sk}^j = \Gamma_{ks}^j$ and $\nabla_i g_{sk} = 0$. If we choose the Hamiltonian depended on functions u^i only, $H = \int h(\mathbf{u}) dx$, then the Poisson bracket (1.6) determines Hydrodynamic type system $u_t^i = w_k^i(\mathbf{u}) u_x^k$, where $w_k^i(\mathbf{u}) = \nabla^i \nabla_k h$ (see [3]). Moreover, we can find "flat coordinates" a^ν (annihilators of the Poisson bracket (1.6), or Casimirs), where all $\Gamma_{\beta\gamma}^\alpha \equiv 0$ and $\bar{g}^{\alpha\beta}$ is constant symmetric nondegenerated metric

$$a_t^\alpha = \partial_x [\bar{g}^{\alpha\beta} \frac{\delta H}{\delta a^\beta}]. \quad (1.7)$$

This Hamiltonian structure allows $(N + 2)$ conservation laws, where first N of them are (1.7), the conservation law of Energy is

$$h_t = \partial_x [\bar{g}^{\alpha\beta} \frac{\partial h}{\partial a^\alpha} \frac{\partial h}{\partial a^\beta}], \quad (1.8)$$

and conservation law of Momentum $P = \int p dx$ is

$$p_t = \partial_x [a^\alpha \frac{\partial h}{\partial a^\alpha} - h], \quad (1.9)$$

where $(\bar{g}_{\alpha\beta} \bar{g}^{\beta\gamma} = \delta_\alpha^\gamma)$

$$p = \frac{1}{2} \bar{g}_{\alpha\beta} a^\alpha a^\beta. \quad (1.10)$$

In more general case $H = \int h(\mathbf{a}, \mathbf{a}_x, \dots, \mathbf{a}_M) dx$ the conservation law of the Momentum is

$$p_t = \partial_x [a^\alpha \frac{\delta H}{\delta a^\alpha} - F], \quad (1.11)$$

where

$$\partial_x F = \frac{\delta H}{\delta a^\alpha} a_x^\alpha. \quad (1.12)$$

and thus

$$F = h - \sum_{n=1}^M (-1)^n a_n^\beta \sum_{k=n}^M (-1)^k \partial_x^{k-n} \frac{\partial h}{\partial a_k^\beta}.$$

In this case formula (1.8) are transformed into more general

$$h_t = \partial_x (\bar{g}^{\alpha\beta} [\sum_{m=1}^M \sum_{k=0}^{m-1} (-1)^k \partial_x^k (\frac{\partial h}{\partial a_m^\alpha}) \partial_x^{m-k} (\frac{\delta H}{\delta a^\beta}) + \frac{1}{2} \frac{\delta H}{\delta a^\alpha} \frac{\delta H}{\delta a^\beta}]).$$

It is easily to check that if evolutionary system (1.1) has $(N+1)$ conservation law densities connected by constraint (1.10), then this system has local Hamiltonian structure (1.7). Also, the evolutionary system (1.7) has the Lagrangian representation

$$S = \int [\frac{1}{2} \bar{g}_{\alpha\beta} \varphi_x^\alpha \varphi_t^\beta - h(\varphi_x, \varphi_{xx}, \dots)] dx dt, \quad (1.13)$$

where $a^\alpha = \varphi_x^\alpha$.

A more complicated case was studied by E.V.Ferapontov and O.I.Mokhov in the article [4] in 1990:

$$\{u^i(z), u^j(z')\} = [g^{ij}(\mathbf{u}(z)) \partial_z - g^{is} \Gamma_{sk}^j u_z^k + \varepsilon u_z^i \partial_z^{-1} u_z^j] \delta(z - z'), \quad (1.14)$$

where $g^{ij}(\mathbf{u})$ is nondegenerated symmetric metric with constant curvature ε (see (1.6)). However, some problems have been unsolved. In this article we present:

1. Canonical coordinates for evolutionary systems with nonlocal Hamiltonian structure determined by the Poisson bracket (1.6). Thus, Hamiltonian structure will be written in compact form (see for comparison (1.7)).
2. The Metric and the Momentum in canonical coordinates (see for comparison (1.9) and (1.10)).
3. The Lagrangian representation (see for comparison (1.13)).
4. Reciprocal transformations connecting Poisson brackets (1.6) and (1.14).
5. The fourth (nonlocal) Hamiltonian structure associated with metric of constant curvature for the Kaup-Boussinesq system.

Local linear-degenerated Lagrangians (Lagrangians are linear with respect to derivatives of t) were studied in [8]. It means that symplectic structure is local (determined by differential operator of arbitrary order), of course, corresponding Hamiltonian structure is

nonlocal, but not invertible in compact form (it means that in general case corresponding differential operator has infinite set of elements). Arbitrary nonlocal Hamiltonian structure has corresponding nonlocal symplectic structure. Moreover this symplectic structure has infinite set of elements too. It is astonishing that namely in case of constant curvature metric nonlocal Hamiltonian structure has local corresponding symplectic structure. By another words, not every nonlocal Hamiltonian structure has inverse local symplectic structure. At first the general reciprocal transformation connecting Poisson brackets (1.6) and (1.14) was presented by E.V.Ferapontov (see below). However, here we present one very special case: one-parametric family of constant curvature metrics ε -intimated to flat case. It means that if in below presented reciprocal transformation anyone put $\varepsilon = 0$, then it will be identical transformation (Moreover in general case recalculation of all attributes for Poisson bracket associated with metric of constant curvature (annihilators, momentum and Hamiltonian) is very complicated problem not solved now. Just in our particular case it was solved and presented below). The metric of constant curvature is well known, however annihilators and momentum for corresponding Poisson brackets were not known as well as Lagrangian representations. Our major aim is construction of Lagrangian representations without constraints for nonlocal Hamiltonian structures. Here we establish a Lagrangian representation for nonlocal Hamiltonian structure associated with differential-geometric Poisson bracket of the first order with metric of constant curvature. This is the first nontrivial example of nonlocal Hamiltonian structures generalizing the local one. Lagrangian representations for Poisson brackets associated with metrics of constant curvature have been obtained by application of special reciprocal transformation (see below) for Lagrangian representations of Poisson brackets associated with metrics of zero curvature. It is amazing, that it is possible. Usually it is not valid. If anyone try to apply arbitrary reciprocal transformation for arbitrary Lagrangian density (which is 2-form), then obtained new Lagrangian representation will not create system connected by abovementioned reciprocal transformation with initial system determined by initial Lagrangian representation. It means that we do not know all Lagrangian representations convertible under reciprocal transformation into others. However, namely in case of constant curvature metric this problem is successfully solved in this article. Moreover, we would like to emphasize that in case of constant curvature knowledge of annihilators and momentum is not enough for direct reconstruction of Lagrangian representation with respect to Hamiltonian structures associated with metrics of zero curvature. This is nontrivial problem are solved by specific choice of annihilators (special N from all $N + 1$, see below). This article contains several Sections. In Section II we formulate a theorem about canonical coordinates (Casimirs or annihilators of Poisson brackets), where this nonlocal Hamiltonian structure will be compactly presented. In Section III we present two theorems about relationship between this nonlocal and local Hamiltonian structures. In the Section IV we present two remarkable examples, which allow this nonlocal Hamiltonian structure. One of them is the Calogero KdV equation related to the KdV equation by the combination of differential substitutions, another is the Thrice-Modified Kaup-Boussinesq system which is related to the Kaup-Boussinesq system by a combination of differential substitutions. In Section V we establish a Lagrangian representation for an arbitrary evolutionary system with this nonlocal Hamiltonian structure. And we show that canonical coordinates presented in Section II determine potential functions in this Lagrangian rep-

resentation. Moreover, we show relationship between this Lagrangian representation (for evolutionary system with nonlocal Hamiltonian structure) and with Lagrangian representation for evolutionary system determined by local Hamiltonian structure. In Section VI we describe a very important example of the first four Hamiltonian structures of the Kaup-Boussinesq system. We demonstrate validity of infinite-dimensional analog of Darboux theorem for this Hamiltonian structures by straightforward calculations, where every Hamiltonian structure can be presented in their canonical form "d/dx". In all cases we present Lagrangian representations, describe relationships between all formulas, and present a new integrable evolutionary system connected with Thrice-Modified Kaup-Boussinesq system by reciprocal transformation, which has local Hamiltonian structure reduced from nonlocal Hamiltonian structure of aforementioned type.

To this moment we know many integrable systems possessing this nonlocal Hamiltonian structure. We mention here just some famous of them. These are Korteweg-de Vries equation, Kaup-Boussinesq system, Multi-component Long-Short Wave Resonance (see articles of Najima & Oikawa and Melnikov), Coupled KdV (see articles of Antonowicz & Fordy) and so on. Moreover, averaged integrable systems are hydrodynamic type systems (see articles of Dubrovin & Novikov), which possess the same type of Hamiltonian structures. The modern level of development of Hamiltonian structures (see below) needs for introducing them into other areas of scientific creation like fields theory, theory of instability in fluid mechanics and ets. We hope that presented results can be interesting for specialists not working in theory of integrable systems or in differential geometry as well.

2 Canonical Coordinates for the Metrics of Constant Curvature

Theorem 1 *The evolutionary system (1.1) (see (1.14))*

$$u_y^i = [g^{ij}\partial_z - g^{is}\Gamma_{sk}^j u_z^k + \varepsilon u_z^i \partial_z^{-1} u_z^j] \frac{\delta H}{\delta u^j}, \quad i = 1, 2 \dots N \quad (2.1)$$

has

1. *Casimir functionals $H_\alpha = \int c^\alpha(\mathbf{u})dz$, where $\alpha = 1, 2 \dots N$ (annihilators of the Poisson bracket (1.14)), which are determined by (see (2.1))*

$$[g^{ij}\partial_z - g^{is}\Gamma_{sk}^j u_z^k + \varepsilon u_z^i \partial_z^{-1} u_z^j] \frac{\delta H_\alpha}{\delta u^j} = 0, \quad (2.2)$$

or by

$$\partial_{ik} c^\alpha - \Gamma_{ik}^n \partial_n c^\alpha + \varepsilon g_{ik} c^\alpha = 0. \quad (2.3)$$

(the system (2.3) has $(N + 1)$ solutions. Any N of them are functionally independent.)

2. The metric $g^{\alpha\beta}$ in Casimirs $c^\alpha(u)$ (see (2.3)) is

$$g^{\alpha\beta} = \bar{g}^{\alpha\beta} - \varepsilon c^\alpha c^\beta, \quad (2.4)$$

where $\bar{g}^{\alpha\beta}$ is nondegenerated symmetric constant matrix. The metric $g_{\alpha\beta}$ (see (2.4)), where $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$ is

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \varepsilon \frac{\bar{g}_{\alpha\gamma} c^\gamma \bar{g}_{\beta\nu} c^\nu}{1 - \varepsilon \bar{g}_{\gamma\nu} c^\gamma c^\nu}, \quad (2.5)$$

where $\bar{g}_{\alpha\beta} \bar{g}^{\beta\gamma} = \delta_\alpha^\gamma$. The Christoffel symbols are $\Gamma_{\beta\gamma}^\alpha = \varepsilon g_{\beta\gamma} c^\alpha$.

3. The first N conservation laws (see (1.12) and (2.4)) are

$$c_y^\alpha = \partial_z [g^{\alpha\beta} \frac{\delta H}{\delta c^\beta} + \varepsilon c^\alpha F], \quad \alpha = 1, 2, \dots, N \quad (2.6)$$

4. The conservation law of the momentum

$$p_y = \partial_z [(1 - \varepsilon p)(c^\alpha \frac{\delta H}{\delta c^\alpha} - F)], \quad (2.7)$$

where the Momentum $P = \int p(\mathbf{c}) dx$ is

$$p = \frac{1}{\varepsilon} [1 - \sqrt{1 - \varepsilon \bar{g}_{\alpha\beta} c^\alpha c^\beta}], \quad (2.8)$$

which is determined by (2.6)

$$g^{\alpha\beta} \partial_\beta p + \varepsilon c^\alpha p = c^\alpha. \quad (2.9)$$

5. The conservation law of the Energy is

$$h_y = \partial_z (\frac{1}{2} g^{\alpha\beta} \frac{\partial h}{\partial c^\alpha} \frac{\partial h}{\partial c^\beta} + \frac{\varepsilon}{2} h^2) \quad (2.10)$$

for hydrodynamic type systems ($H = \int h(\mathbf{c}) dz$) or in more general case (see (2.6))

$$h_y = \partial_z (\sum_{m=1}^M \sum_{k=0}^{m-1} (-1)^k \partial_z^k (\frac{\partial h}{\partial c_m^\alpha}) \partial_z^{m-k} (g^{\alpha\beta} \frac{\delta H}{\delta c^\beta} + \varepsilon c^\alpha F) + \frac{1}{2} g^{\alpha\beta} \frac{\delta H}{\delta c^\alpha} \frac{\delta H}{\delta c^\beta} + \frac{\varepsilon}{2} F^2),$$

$$\text{where } H = \int h(\mathbf{c}, \mathbf{c}_z, \dots, \mathbf{c}_M) dz \text{ and } F = h - \sum_{n=1}^M (-1)^n c_n^\beta \sum_{k=n}^M (-1)^k \partial_x^{k-n} \frac{\partial h}{\partial c_k^\beta}.$$

Remark I: If $H = \int h(\mathbf{c}) dz$, then evolutionary system (2.1) transforms into Hydrodynamic type system $u_y^i = w_k^i(\mathbf{u}) u_z^k$, where $w_k^i(\mathbf{u}) = \nabla^i \nabla_k h + \varepsilon h \delta_k^i$ (see [4]).

Remark II: If $\varepsilon \rightarrow 0$, all formulas (2.1-10) transform into the "flat" case (1.6-11).

Proof: can be obtained by straightforward calculation.

Example: Hydrodynamic type systems possessing nonlocal Hamiltonian structure (2.1) associated with elliptic coordinates were described in [5], where all exact formulas (Casimirs, metrics, conservation law densities) were presented too.

Theorem 2 *If evolutionary system (1.1) has $(N+1)$ conservation law densities connected by constraint (2.8), then this system has nonlocal Hamiltonian structure associated with metric of constant curvature.*

Proof: We take evolutionary system (1.1) in divergent form $c_y^\alpha = \partial_z b^\alpha(\mathbf{c}, \mathbf{c}_z, \mathbf{c}_{zz}, \dots)$, then additional conservation law $p_y = \partial_z b(\mathbf{c})$ yields relationship $\partial_z b(\mathbf{c}) = \bar{g}_{\alpha\beta} \frac{c^\beta}{1-\varepsilon p} b_z^\alpha$. It is valid if and only if $\bar{g}_{\alpha\beta} b^\beta = \delta S / \delta q^\alpha$, where $q^\alpha = c^\alpha / (1 - \varepsilon p)$ and $S = \int s(\mathbf{q}, \mathbf{q}_z, \mathbf{q}_{zz}, \dots) dz$. It means that $p_y = \partial_z [q^\beta \frac{\delta S}{\delta q^\beta} - R]$ and $c_y^\alpha = \partial_z [\bar{g}^{\alpha\beta} \frac{\delta S}{\delta q^\beta}]$, where $\partial_z R = \frac{\delta S}{\delta q^\alpha} q_z^\alpha$. Since $\frac{\delta S}{\delta q^\alpha} = (1 - \varepsilon p) [\frac{\delta S}{\delta c^\alpha} - \varepsilon c^\gamma \frac{\delta S}{\delta c^\gamma} \bar{g}_{\alpha\beta} c^\beta]$, anyone can immediately obtain (2.6) and (2.7), where $h = (1 - \varepsilon p)s$.

3 Reciprocal Transformation and nonlocal Hamiltonian structures.

In this Section we establish canonical reciprocal transformation between local Hamiltonian structure (1.7) and nonlocal Hamiltonian structure (2.6).

The general reciprocal transformation between Hamiltonian structures (1.7) and (2.6) was constructed in [6] by E.Ferapontov in 1995 for Hydrodynamic type systems

$$u_t^i = v_k^i(\mathbf{u}) u_x^k \quad \text{and} \quad u_y^i = w_k^i(\mathbf{u}) u_z^k. \quad (3.1)$$

If the first system in (3.1) has local Hamiltonian structure (see (1.7-10)), then we can introduce the reciprocal transformation

$$dy = A(\mathbf{u})dx + B(\mathbf{u})dt, \quad dz = C(\mathbf{u})dx + D(\mathbf{u})dt, \quad (3.2)$$

where

$$\begin{aligned} A(\mathbf{u}) &= \alpha h + \beta p + \gamma_\nu a^\nu + \zeta, & B(\mathbf{u}) &= \frac{\alpha}{2} \bar{g}^{\mu\nu} h_\mu h_\nu + \beta(a^\nu h_\nu - h) + \gamma_\nu \bar{g}^{\nu\mu} h_\mu + \eta \\ C(\mathbf{u}) &= \bar{\alpha} h + \bar{\beta} p + \bar{\gamma}_\nu a^\nu + \bar{\zeta}, & D(\mathbf{u}) &= \frac{\bar{\alpha}}{2} \bar{g}^{\mu\nu} h_\mu h_\nu + \bar{\beta}(a^\nu h_\nu - h) + \bar{\gamma}_\nu \bar{g}^{\nu\mu} h_\mu + \bar{\eta}, \end{aligned}$$

and $\alpha, \beta, \gamma_\nu, \zeta, \eta, \bar{\alpha}, \bar{\beta}, \bar{\gamma}_\nu, \bar{\zeta}, \bar{\eta}$ are arbitrary constants.

Theorem 3 ([6]) *The Hydrodynamic type system (x, t) with local Hamiltonian structure (1.7) transforms into the Hydrodynamic type system (y, z) with nonlocal Hamiltonian structure (2.6) if*

$$\bar{g}^{\mu\nu} \gamma_\mu \gamma_\nu - 2\alpha\eta - 2\beta\zeta = \varepsilon, \quad (3.3)$$

$$\bar{g}^{\mu\nu} \bar{\gamma}_\mu \bar{\gamma}_\nu = 2\bar{\alpha}\bar{\eta} + 2\bar{\beta}\bar{\zeta}, \quad \bar{g}^{\mu\nu} \gamma_\mu \bar{\gamma}_\nu = \alpha\bar{\eta} + \bar{\alpha}\eta + \beta\bar{\zeta} + \bar{\beta}\zeta$$

By choosing special constants in (3.3) we present particular, but more simple and more clear

Theorem 4 *The evolutionary system (x, t) with local Hamiltonian structure (1.7) transforms itself into the evolutionary system (y, z) with nonlocal Hamiltonian structure (2.6) by the reciprocal transformation*

$$dy = dt, \quad dz = (1 + \frac{\varepsilon}{2}p)dx + \frac{\varepsilon}{2}qdt, \quad (3.4)$$

where $\partial_t p = \partial_x q$ and $q = a^\alpha \frac{\delta H}{\delta a^\alpha} - F$ (see (1.11) and (1.12)). Then

$$\begin{aligned} 1. \quad \bar{h}(\mathbf{c}, \mathbf{c}_z, \dots) &= h(\mathbf{a}, \mathbf{a}_x, \dots) / (1 + \frac{\varepsilon}{2}p), \\ 2. \quad \bar{p} &= p / (1 + \frac{\varepsilon}{2}p), \\ 3. \quad c^\alpha &= a^\alpha / (1 + \frac{\varepsilon}{2}p), \\ 4. \quad g^{\alpha\beta} &= \bar{g}^{\alpha\beta} - \varepsilon c^\alpha c^\beta, \\ 5. \quad c_y^\alpha &= \partial_z [g^{\alpha\beta} \frac{\delta \bar{H}}{\delta c^\beta} + \varepsilon c^\alpha \bar{F}], \end{aligned}$$

where $\partial_x = (1 + \frac{\varepsilon}{2}p)\partial_z$, $\bar{H} = \int \bar{h}(\mathbf{c}, \mathbf{c}_z, \dots) dz$ and $\partial_z \bar{F} = \frac{\delta \bar{H}}{\delta c^\beta} c_z^\beta$

Remark I: if $\varepsilon \rightarrow 0$, then all this formulas transform into local case (see Section I). E.V.Ferapontov have studied another particular case too

$$dy = dt, \quad dz = (p + \frac{1}{2})dx + (a^\alpha \frac{\partial h}{\partial a^\alpha} - h)dt.$$

However, just in our case we describe one-parameter (ε - parameter) family of metrics of constant curvature, where if $\varepsilon = 0$ (3.4) is identical. In our case we present by above theorem recalculation for annihilators, momentum and Hamiltonian – all that was absent in earlier articles.

Remark II: The conditions 2. and 3. yield relationship between (2.8) and (1.10). The inverse formulas are $p = \bar{p} / (1 - \frac{\varepsilon}{2}\bar{p})$, $h = \bar{h} / (1 - \frac{\varepsilon}{2}\bar{p})$, $a^\alpha = c^\alpha / (1 - \frac{\varepsilon}{2}\bar{p})$.

Proof: An arbitrary conservation law for the evolutionary system (1.7) can be presented in its divergent form $d\xi = hdx + fdt$. If we apply the reciprocal transformation (3.4) for all $(N+2)$ conservation laws (1.7-9) and (1.11), then we at once obtain conditions of this theorem.

4 Remarkable examples

1. It is well-known fact (see for instance [7]) that **the Calogero Korteweg-de Vries equation** (CKdV)

$$u_y = \partial_z [u_{zz} + \frac{3}{2u}(1 - u^2)] \quad (4.1)$$

has the nonlocal Hamiltonian structure (1.14) (see (2.1))

$$u_y = [u^2 \partial_z + uu_z - u_z \partial_z^{-1} u_z] \frac{\delta H}{\delta u} \quad (4.2)$$

where $H = -\frac{1}{2} \int \frac{1+u^2}{u^2} dz$. However, here $\bar{g}^{11} \equiv 0$. Thus, the CKdV equation has extraordinary momentum $P = \int 1 \cdot dz$ and two Casimirs $Q_1 = \int u dz$ and $Q_2 = \int dz/u$. Here we introduce other particular reciprocal transformation (see (3.2), (3.3) and (4.1))

$$dt = dy, \quad dx = u dz + [u_{zz} + \frac{3}{2u}(1 - u_z^2)] dy. \quad (4.3)$$

Then inverse reciprocal transformation is

$$dy = dt, \quad dz = w dx + [\frac{w_{xx}}{w^3} - \frac{3w_x^2}{2w^4} - \frac{3}{2}w^2] dt. \quad (4.4)$$

where $uw = 1$ and $\partial_z = u \partial_x$, and the CKdV equation transforms into

$$w_t = \partial_x [\frac{w_{xx}}{w^3} - \frac{3w_x^2}{2w^4} - \frac{3}{2}w^2] \quad (4.5)$$

This equation (4.5) has local Hamiltonian structure

$$w_t = \partial_x \frac{\delta \bar{H}}{\delta w} \quad (4.6)$$

where $(\bar{H} = \int \bar{h}(w, w_x) dx = \int \bar{h}(1/u, (1/u)_z/u) u dz = \int h(u, u_z) dz)$ the Hamiltonian is $\bar{H} = -\frac{1}{2} \int [\frac{w_x^2}{w^3} + w^3] dx$, the Momentum is $\bar{P} = \frac{1}{2} Q_2 = \frac{1}{2} \int dz/u = \frac{1}{2} \int w^2 dx$, the Casimir is $\bar{Q} = P = \int 1 \cdot dz = \int w dx$ and other Casimir for the nonlocal Hamiltonian structure (4.2) transforms into trivial Casimir $Q_1 = \int u dz = \int 1 \cdot dx$.

Here we introduce potential function z (see (1.13) and (4.4)), then the equation (4.5) has the Lagrangian representation

$$S = \frac{1}{2} \int [z_x z_t + \frac{z_{xx}^2}{z_x^3} + z_x^3] dx dt \quad (4.7)$$

where $w = z_x$. We can apply the reciprocal transformation (4.4) for the 2-form

$$\Omega = [z_x z_t + \frac{z_{xx}^2}{z_x^3} + z_x^3] dx \wedge dt.$$

Then this 2-form

$$\Omega = [we + \frac{w_x^2}{w^3} + w^3] dx \wedge dt,$$

where $e = z_t$ transforms into

$$\Omega = [\frac{e}{u} + \frac{1+u_z^2}{u^3}] u dz \wedge dy,$$

where $dx \wedge dt = u dz \wedge dy$ and $\partial_x = w \partial_z$ (see (4.3) and (4.4)). Thus, this 2-form

$$\Omega = \left[-\frac{x_y}{x_z} + \frac{1 + x_{zz}^2}{x_z^2} \right] dz \wedge dy,$$

where $u = x_z$ and $e = -x_y/x_z$ (see (4.3)) yields the Lagrangian representation for the CKdV equation

$$S = \frac{1}{2} \int \left[-\frac{x_y}{x_z} + \frac{1 + x_{zz}^2}{x_z^2} \right] dz dy, \quad (4.8)$$

Thus, we have described a relationship between a Lagrangian representation for evolutionary equation with local Hamiltonian structure (4.6) and a Lagrangian representation for evolutionary equation with nonlocal Hamiltonian structure (4.2). This action (4.8) have been established in article [8] for the Krichever-Novikov equation. However, here we will give a generalization of this Lagrangian representation on N -component case for evolutionary systems with nonlocal Hamiltonian structure (2.6).

2. The Thrice-Modified Kaup-Boussinesq system (see [9])

$$c_y = \partial_z \left[-\frac{1}{2} b(1+b^2) + \varepsilon (cb_z - bc_z) \right], \quad b_y = \partial_z \left[-\frac{(1-b^2)^2}{2c} + \varepsilon (bb_z + \frac{1-b^2}{c} c_z) \right] \quad (4.9)$$

(ε is arbitrary constant, not curvature here!) has nonlocal *Hamiltonian* structure

$$\begin{aligned} c_y &= \frac{1}{4} \partial_z \left[bc \frac{\delta H}{\delta b} + c^2 \frac{\delta H}{\delta c} - cF \right] \\ b_y &= \frac{1}{4} \partial_z \left[(b^2 - 1) \frac{\delta H}{\delta b} + bc \frac{\delta H}{\delta c} - bF \right], \end{aligned} \quad (4.10)$$

where the Hamiltonian is $H = 2 \int [b(1 - b^2) - 2\varepsilon bc_z] dz/c$ and $\partial_z F = \frac{\delta H}{\delta b} b_z + \frac{\delta H}{\delta c} c_z$. This bracket is determined by the differential-geometric *Poisson* bracket with metric of constant curvature (1.14)

$$\begin{aligned} \{b(z), b(z')\} &= \frac{1}{4} [(b^2 - 1) \partial_z + bb_z - b_z \partial_z^{-1} b_z] \delta(z - z') \\ \{b(z), c(z')\} &= \frac{1}{4} [cb \partial_z + cb_z - b_z \partial_z^{-1} c_z] \delta(z - z') \\ \{c(z), b(z')\} &= \frac{1}{4} [bc \partial_z + bc_z - c_z \partial_z^{-1} b_z] \delta(z - z') \\ \{c(z), c(z')\} &= \frac{1}{4} [c^2 \partial_z + cc_z - c_z \partial_z^{-1} c_z] \delta(z - z'). \end{aligned} \quad (4.11)$$

Since, just $\bar{g}^{11} = -1$ ($\bar{g}^{12} = \bar{g}^{21} = \bar{g}^{22} = 0$, e.g. $\det \bar{g}^{\alpha\beta} = 0$) the system (4.13)

has extraordinary momentum $P = \int 1 \cdot dz$ (compare with (2.4-2.6) and (2.8)), but three Casimirs

$$Q_1 = \int \frac{1-b^2}{c} dz, \quad Q_2 = \int b dz, \quad Q_3 = \int c dz.$$

These conservation law densities determine the constraint

$$q_1 q_3 + q_2^2 = 1 \quad (4.12)$$

where $Q_\alpha = \int q_\alpha dz$, $\alpha = 1, 2, 3$. The Poisson bracket (4.11) can be reduced into canonical form (1.7) by multi-parameter reciprocal transformation (3.2) (see (3.3)). Here we can for instance use simplest particular reciprocal transformation

$$dt = dy, \quad dx = cdz + \left[-\frac{1}{2}b(1+b^2) + \varepsilon(cb_z - bc_z)\right]dy \quad (4.13)$$

(see the first equation in (4.9)). Then, the inverse reciprocal transformation is

$$dy = dt, \quad dz = udx + \frac{1}{2}\left[w + \frac{w^3}{u^2} - 2\varepsilon\frac{w_x}{u^2}\right]dt, \quad (4.14)$$

where $uc = 1$, $b = wc$ and $\partial_z = c\partial_x$ or $\partial_x = u\partial_z$. And we at once obtain other integrable system

$$u_t = \frac{1}{2}\partial_x\left[w + \frac{w^3}{u^2} - 2\varepsilon\frac{w_x}{u^2}\right], \quad w_t = \frac{1}{2}\partial_x\left[3\frac{w^2}{u} - u - 2\varepsilon\frac{u_x}{u^2}\right], \quad (4.15)$$

which has local Hamiltonian structure

$$u_t = \frac{1}{4}\partial_x\frac{\delta\bar{H}}{\delta u}, \quad w_t = -\frac{1}{4}\partial_x\frac{\delta\bar{H}}{\delta w} \quad (4.16)$$

with $(\bar{H} = \int \bar{h}(b, c, c_z) dz = \int \bar{h}(w/u, 1/u, (1/u)_z/u) u dx = \int h(u, w, w_x) dx)$ the Hamiltonian $\bar{H} = 2 \int [uw - \frac{w^3 - 2\varepsilon w_x}{u}] dx$, also with Momentum $\bar{P} = 2Q_1 = 2 \int q_1 dz = 2 \int (u^2 - w^2) dx$ (see (4.12)), two Casimirs $\bar{Q}_2 = Q_2 = \int b dz = \int w dx$, $\bar{Q}_3 = P = \int 1 \cdot dz = \int u dx$ and other Casimir for the nonlocal Hamiltonians structure (4.10) transforms into trivial Casimir $Q_3 = \int c dz = \int 1 \cdot dx$.

The evolutionary system (4.16) with the Poisson bracket

$$\{u(x), u(x')\} = -\{w(x), w(x')\} = \frac{1}{4}\delta'(x - x') \quad (4.17)$$

has the Lagrangian representation (see (1.13) and (4.14))

$$S = \frac{1}{2} \int [z_x z_t - \varphi_x \varphi_t - \frac{2\varepsilon \varphi_{xx} - \varphi_x^3}{z_x} - z_x \varphi_x] dx dt, \quad (4.18)$$

where $w = \varphi_x$. We can apply the reciprocal transformation (4.14) for the 2-form

$$\Omega = [z_x z_t - \varphi_x \varphi_t - \frac{2\varepsilon \varphi_{xx} - \varphi_x^3}{z_x} - z_x \varphi_x] dx \wedge dt.$$

Then this 2-form

$$\Omega = [ue - wv - \frac{2\varepsilon w_x - w^3}{u} - uw] dx \wedge dt,$$

where $e = z_t$ and $v = \varphi_t$, transforms itself into

$$\Omega = [\frac{e}{c} - \frac{bv}{c} - 2\varepsilon(\frac{b}{c})_z + \frac{b^3}{c^2} - \frac{b}{c^2}] cdz \wedge dy,$$

where $dx \wedge dt = cdz \wedge dy$ (see (4.13)). Since $d\varphi = wdx + vdt = bdz + (v - be)dy$, then this 2-form

$$\Omega = [-\frac{1 - \varphi_z^2}{x_z} x_y - \varphi_z \varphi_y - 2\varepsilon x_z (\frac{\varphi_z}{x_z})_z + \frac{\varphi_z^3}{x_z} - \frac{\varphi_z}{x_z}] dz \wedge dy,$$

where $c = x_z$, $e = -x_y/x_z$, $b = \varphi_z$ and $v = \varphi_y - \varphi_z x_y/x_z$ (see (4.13)), yields the Lagrangian representation for the Thrice-Modified Kaup-Boussinesq system

$$S = \frac{1}{2} \int [\frac{1 - \varphi_z^2}{x_z} x_y + \varphi_z \varphi_y - 2\varepsilon \frac{x_{zz}}{x_z} \varphi_z + \frac{\varphi_z(1 - \varphi_z^2)}{x_z}] dz dy. \quad (4.19)$$

Remark: The Kaup-Boussinesq system has nonlocal Hamiltonian structure with de-generated constant $\bar{g}^{\alpha\beta}$ matrix (see (2.4)) in coordinates (q_2, q_3) (see (4.12)). However, this Poisson bracket (4.11) easily can be transformed into canonical form (2.6) with canonical metrics (2.4), if we change variable $q_2 \rightarrow 1 - q$. Then (4.12) at once yields a Momentum (see (2.8))

$$q = 1 - \sqrt{1 - q_1 q_3}$$

expressed in its canonical variables Casimirs q_1 and q_3 (it is easy to see, that the Momentum $P = \int q dz$ (where $q = 1 - q_2$) is a linear combination of two Casimirs Q_3 and Q_2). Immediately we obtain (see (2.8)) all non-zero components of constant matrix $\bar{g}_{\alpha\beta}$: $\bar{g}_{12} = \bar{g}_{21} = 1/2$ and curvature $\varepsilon = 1$. Thus, $\bar{g}^{12} = \bar{g}^{21} = 2$ and the Thrice-Modified Kaup-Boussinesq system (4.9) can be re-written in variables (r, c) (see (2.4) and (2.6)), where $rc + b^2 = 1$ (see (4.12))

$$r_y = \frac{1}{2} \partial_z [\frac{r^2}{\sqrt{1-rc}} + \varepsilon \frac{r^2 c_z + (2-rc)r_z}{\sqrt{1-rc}}], \quad c_y = -\frac{1}{2} \partial_z [(2-rc)\sqrt{1-rc} + \varepsilon \frac{(2-rc)c_z + c^2 r_z}{\sqrt{1-rc}}].$$

This system has nonlocal Hamiltonian structure (2.6)

$$r_y = \frac{1}{4} \partial_z [-r^2 \frac{\delta H}{\delta r} + (2-rc) \frac{\delta H}{\delta c} + rF], \quad c_y = \frac{1}{4} \partial_z [(2-rc) \frac{\delta H}{\delta r} - c^2 \frac{\delta H}{\delta c} + cF],$$

where $H = 2 \int \sqrt{1 - rc}(r - 2\epsilon c_z/c) dz$ and $\partial_z F = \frac{\delta H}{\delta r} r_z + \frac{\delta H}{\delta c} c_z$ (see (4.10)). Thus, the Poisson bracket (4.11) determined by Lagrangian representation (4.19) for the evolutionary system (4.9) yields the canonical Poisson bracket (2.4)

$$\begin{aligned}\{r(z), r(z')\} &= \frac{1}{4}[-r^2 \partial_z - r r_z + r_z \partial_z^{-1} r_z] \delta(z - z'), \\ \{r(z), c(z')\} &= \frac{1}{4}[(2 - rc) \partial_z - c r_z + r_z \partial_z^{-1} c_z] \delta(z - z'), \\ \{c(z), r(z')\} &= \frac{1}{4}[(2 - rc) \partial_z - r c_z + c_z \partial_z^{-1} r_z] \delta(z - z'), \\ \{c(z), c(z')\} &= \frac{1}{4}[-c^2 \partial_z - c c_z + c_z \partial_z^{-1} c_z] \delta(z - z').\end{aligned}$$

5 Lagrangian Representation

Major result:

Theorem 5 *The evolutionary system (1.1) with nonlocal Hamiltonian structure (see (2.1)) determined by differential-geometric Poisson bracket of the first order associated with metrics of constant curvature (1.14) has the Lagrangian representation*

$$S = \int \left[\frac{1 - \bar{g}_{\alpha\nu} \varphi_z^\alpha \varphi_z^\nu}{2x_z} x_y + \frac{1}{2} \bar{g}_{\alpha\nu} \varphi_z^\alpha \varphi_y^\nu - h(x_z, \varphi_z, x_{zz}, \varphi_{zz}, \dots) \right] dz dy \quad (5.1)$$

Proof: From variational derivatives $\delta S / \delta \varphi^\alpha = 0$, $\delta S / \delta x = 0$ and compatibility condition $(x_z)_y = (x_y)_z$, respectively, we obtain evolutionary system on $(N+2)$ equations

$$v_y^\alpha + \partial_z(\rho v^\alpha - \bar{g}^{\alpha\nu} \delta H / \delta v^\nu) = 0, \quad w_y + \partial_z(\rho w - \delta H / \delta u) = 0, \quad u_y + \partial_z(\rho u) = 0, \quad (5.2)$$

where $u = x_z$, $\rho = -x_y / x_z$, $v^\alpha = \varphi_z^\alpha$, $H = \int h(u, \mathbf{v}, u_z, \mathbf{v}_z, \dots) dz$ and constraint

$$1 = 2uw + \bar{g}_{\alpha\nu} v^\alpha v^\nu. \quad (5.3)$$

The system (5.2) is over-determined system. Thus, from the obvious condition $\partial_y(2uw + \bar{g}_{\alpha\nu} v^\alpha v^\nu) = 0$ (see (5.3)) we obtain an explicit expression of the function ρ

$$\rho = u \frac{\delta H}{\delta u} + v^\alpha \frac{\delta H}{\delta v^\alpha} - F, \quad (5.4)$$

where $\partial_z F = \frac{\delta H}{\delta v^\alpha} v_z^\alpha + \frac{\delta H}{\delta u} u_z$ (see (1.12)).

At first we introduce new variables (p, q, \mathbf{v}) by

$$u = 1 - p - q, \quad 2w = 1 - p + q, \quad q^\alpha = v^\alpha, \quad (5.5)$$

then all partial derivatives are

$$\frac{\partial h}{\partial u} = -\frac{1-p}{1-p-q} \frac{\partial h}{\partial q}, \quad \frac{\partial h}{\partial v^\alpha} = \frac{\partial h}{\partial q^\alpha} - \frac{\bar{g}_{\alpha\nu} q^\nu}{1-p-q} \frac{\partial h}{\partial q}. \quad (5.6)$$

For simplicity and without loss of generality it is sufficient, if we will study just the hydrodynamic type case, where $H = \int h(u, v) dz$. Then (see (5.4))

$$\rho = (q \frac{\partial h}{\partial q} + q^\alpha \frac{\partial h}{\partial q^\alpha} - h) - \frac{1}{1-p-q} \frac{\partial h}{\partial q} \quad (5.7)$$

and system (5.2) transforms itself into

$$q_y = \partial_z [-\frac{\partial h}{\partial q} - q(q \frac{\partial h}{\partial q} + q^\alpha \frac{\partial h}{\partial q^\alpha} - h)], \quad q_y^\alpha = \partial_z [\bar{g}^{\alpha\nu} \frac{\partial h}{\partial q^\nu} - q^\alpha (q \frac{\partial h}{\partial q} + q^\nu \frac{\partial h}{\partial q^\nu} - h)]. \quad (5.8)$$

This system has the momentum (see (5.3) and (5.5))

$$p = 1 - \sqrt{1 + q^2 - \bar{g}_{\alpha\nu} q^\alpha q^\nu} \quad (5.9)$$

(thus, curvature $\varepsilon = 1$, see (2.8)) and conservation law of momentum is

$$p_y = \partial_z [(1-p)(q \frac{\partial h}{\partial q} + q^\alpha \frac{\partial h}{\partial q^\alpha} - h)]. \quad (5.10)$$

Here we introduce new variables $c^0 = q$, $c^\alpha = q^\alpha$, $\alpha = 1, 2, \dots, N$. Then the system (5.8) is exactly the system (2.6) with constant matrices

$$\bar{g}^{\alpha\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \bar{g}^{\alpha\nu} \end{pmatrix}, \quad \bar{g}_{\alpha\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \bar{g}_{\alpha\nu} \end{pmatrix}. \quad (5.11)$$

Remark: If we introduce the reciprocal transformation (see (5.2))

$$dt = dy, \quad dx = udz - \rho dy \quad (5.12)$$

we can apply (5.12) for 2-form (see (5.1) and Section IV)

$$\Omega = [-\rho \frac{1 - \bar{g}_{\alpha\nu} v^\alpha v^\nu}{2} + \frac{1}{2} \bar{g}_{\alpha\nu} v^\alpha e^\nu - h(u, \mathbf{v}, u_z, \mathbf{v}_z, \dots)] dz \wedge dy,$$

where $e^\alpha = \varphi_y^\alpha$. Then this 2-form (where $dx \wedge dt = udz \wedge dy$, see (5.12))

$$\Omega = [-z_t \frac{z_x^2 - \bar{g}_{\alpha\nu} \varphi_x^\alpha \varphi_x^\nu}{2z_x} + \frac{1}{2z_x} \bar{g}_{\alpha\nu} \varphi_x^\alpha (\varphi_t^\nu z_x - \varphi_x^\nu z_t) - z_x h] dx \wedge dt,$$

where $z_x = 1/u$, $z_t = \rho$, $v^\alpha = \varphi_x^\alpha / z_x$, $e^\alpha = \varphi_t^\alpha - z_t \varphi_x^\alpha / z_x$, determines the action (see (1.13))

$$S = \int [-\frac{1}{2} z_x z_t + \frac{1}{2} \bar{g}_{\alpha\nu} \varphi_x^\alpha \varphi_t^\nu - \bar{h}(z_x, \boldsymbol{\varphi}_x, z_{xx}, \boldsymbol{\varphi}_{xx}, \dots)] dx dt \quad (5.12)$$

for the evolutionary system (1.1) with local Hamiltonian structure (1.7) and constant metrics (5.11), where (see (1.7)) $a^0 = z_x$, $a^\alpha = \varphi_x^\alpha$ ($\alpha = 1, 2, \dots, N$) and $\bar{h} = a^0 h$ ($H = \int h dz = \int \bar{h} dx = \int a^0 h dx$).

6 Kaup-Boussinesq system and its nonlocal Hamiltonian structure

Many different integrable systems have different local and nonlocal Hamiltonian structures. For example the Korteweg-de Vries equation has two local Hamiltonian structures and all others are nonlocal. The Kaup-Boussinesq system (see for instance [9])

$$v_y = \partial_z \left[\frac{1}{2} v^2 + \eta \right], \quad \eta_y = \partial_z [v\eta + \varepsilon^2 v_{zz}], \quad (6.1)$$

has the three local Hamiltonian structures determined by following Poisson brackets

$$\{v, \eta\}_1 = \{\eta, v\}_1 = \delta'(z - z'), \quad (6.2)$$

$$\{v, v\}_2 = \delta'(z - z'), \quad \{v, \eta\}_2 = \frac{1}{2}(v\partial_z + v_z)\delta(z - z'), \quad (6.3)$$

$$\{\eta, v\}_2 = \frac{1}{2}v\delta'(z - z'), \quad \{\eta, \eta\}_2 = \varepsilon^2 \delta'''(z - z') + (\eta\partial_z + \frac{1}{2}\eta_z)\delta(z - z'),$$

$$\{v, v\}_3 = (v\partial_z + \frac{1}{2}v_z)\delta(z - z'), \quad (6.4)$$

$$\{v, \eta\}_3 = \varepsilon^2 \delta'''(z - z') + \frac{1}{4}[(v^2 + 4\eta)\partial_z + (v^2 + 2\eta)_z]\delta(z - z'),$$

$$\{\eta, v\}_3 = \varepsilon^2 \delta'''(z - z') + \frac{1}{4}[(v^2 + 4\eta)\partial_z + 2\eta_z]\delta(z - z'),$$

$$\{\eta, \eta\}_3 = \frac{\varepsilon^2}{2}[2v\partial_z^3 + 3v_z\partial_z^2 + 3v_{zz}\partial_z + v_{zzz}]\delta(z - z') + [v\eta\partial_z + \frac{1}{2}(v\eta)_z]\delta(z - z')$$

and nonlocal Hamiltonian structures, where the first of them is

$$\begin{aligned} \{v, v\}_4 &= \varepsilon^2 \delta'''(z - z') + \frac{1}{4}[(3v^2 + 4\eta)\partial_z + \frac{1}{2}(3v^2 + 4\eta)_z - v_z\partial_z^{-1}v_z]\delta(z - z'), \\ \{v, \eta\}_4 &= \frac{\varepsilon^2}{2}[3v\partial_z^3 + 4v_z\partial_z^2 + 3v_{zz}\partial_z + v_{zzz}]\delta(z - z') + \frac{1}{4}[(6v\eta + \frac{1}{2}v^3)\partial_z + \\ &\quad + (\frac{3}{2}v^2v_z + 4\eta v_z + 3v\eta_z) - v_z\partial_z^{-1}\eta_z]\delta(z - z'), \\ \{\eta, v\}_4 &= \frac{\varepsilon^2}{2}[3v\partial_z^3 + 5v_z\partial_z^2 + 4v_{zz}\partial_z + v_{zzz}]\delta(z - z') + \frac{1}{4}[(6v\eta + \frac{1}{2}v^3)\partial_z + \\ &\quad + (2\eta v_z + 3v\eta_z) - \eta_z\partial_z^{-1}v_z]\delta(z - z'), \\ \{\eta, \eta\}_4 &= \varepsilon^4 \delta^V(z - z') + \frac{\varepsilon^2}{4}[(8\eta + 3v^2)\partial_z^3 + \frac{3}{2}(8\eta + 3v^2)_z\partial_z^2 + \\ &\quad + (8\eta + 3v^2)_{zz} + 3vv_{zz}]\partial_z + [(2\eta + v^2)_{zz} + vv_{zz} - \frac{1}{2}v_z^2]_z]\delta(z - z') + \\ &\quad + \frac{1}{4}[(4\eta^2 + 3v^2\eta)\partial_z + \frac{1}{2}(4\eta^2 + 3v^2\eta)_z - \eta_z\partial_z^{-1}\eta_z]\delta(z - z'). \end{aligned} \quad (6.5)$$

The first Miura transformation

$$\eta = (v^2 - a^2)/4 - \varepsilon a_z \quad (6.6)$$

connects the Kaup-Boussinesq system (6.1) and the Modified Kaup-Boussinesq system

$$a_y = \partial_z \left[\frac{1}{2} v a - \varepsilon v_z \right], \quad v_y = \partial_z \left[\frac{1}{4} (3v^2 - a^2) - \varepsilon a_z \right], \quad (6.7)$$

which has two local Hamiltonian structures determined by the Poisson brackets

$$\{a, a\}_1 = -\delta'(z - z'), \quad \{v, v\}_1 = \delta'(z - z'), \quad (6.8)$$

$$\{a, a\}_2 = 0, \quad \{a, v\}_2 = -\varepsilon \delta''(z - z') + \frac{1}{2} (a \partial_z + a_z) \delta(z - z'), \quad (6.9)$$

$$\{v, a\}_2 = \varepsilon \delta''(z - z') + \frac{1}{2} a \delta'(z - z'), \quad \{v, v\}_2 = (v \partial_z + \frac{1}{2} v_z) \delta(z - z')$$

and nonlocal Hamiltonian structures, where the first of them is

$$\{a, a\}_3 = -\varepsilon^2 \delta'''(z - z') + \frac{1}{4} [a^2 \partial_z + a a_z - a_z \partial_z^{-1} a] \delta(z - z'), \quad (6.10)$$

$$\{a, v\}_3 = \left[-\frac{\varepsilon}{2} (2v \partial_z^2 + 3v_z \partial_z + v_{zz}) + \frac{1}{4} (2av \partial_z + av_z + 2va_z - a_z \partial_z^{-1} v) \right] \delta(z - z'),$$

$$\{v, a\}_3 = \varepsilon (v \partial_z + \frac{1}{2} v_z) \delta'(z - z') + \frac{1}{4} [2av \partial_z + av_z - v_z \partial_z^{-1} a] \delta(z - z'),$$

$$\{v, v\}_3 = \varepsilon^2 \delta'''(z - z') + \frac{1}{4} [(3v^2 + 4\eta) \partial_z + \frac{1}{2} (3v^2 + 4\eta)_z - v_z \partial_z^{-1} v] \delta(z - z').$$

The second Miura transformation

$$v = a b + 2\varepsilon b_z \quad (6.11)$$

connects the Modified Kaup-Boussinesq system (6.7) and the Twice-Modified Kaup-Boussinesq system

$$b_y = \frac{1}{2} \partial_z [a(b^2 - 1) + 2\varepsilon b b_z], \quad a_y = \frac{1}{2} \partial_z [a^2 b - 2\varepsilon b a_z - 4\varepsilon^2 b_{zz}], \quad (6.12)$$

which has one local Hamiltonian structure determined by the Poisson bracket

$$\{b, a\}_1 = \frac{1}{2} \delta'(z - z'), \quad \{a, b\}_1 = \frac{1}{2} \delta'(z - z') \quad (6.13)$$

and nonlocal Hamiltonian structures, where the first of them is

$$\{b, b\}_2 = \frac{1}{4} [(b^2 - 1) \partial_z + b b_z - b_z \partial_z^{-1} b] \delta(z - z'), \quad (6.14)$$

$$\{b, a\}_2 = \frac{\varepsilon}{2} (b \partial_z + b_z) \delta'(z - z') + \frac{1}{4} [ab \partial_z + ab_z - b_z \partial_z^{-1} a] \delta(z - z'),$$

$$\{a, b\}_2 = -\frac{\varepsilon}{2} (b \partial_z + b_z) \delta'(z - z') + \frac{1}{4} [ab \partial_z + ba_z - a_z \partial_z^{-1} b] \delta(z - z'),$$

$$\{a, a\}_2 = -\varepsilon^2 \delta'''(z - z') + \frac{1}{4} [a^2 \partial_z + a a_z - a_z \partial_z^{-1} a] \delta(z - z'),$$

The third Miura transformation (see for comparison (4.12))

$$ac + b^2 + 2\varepsilon c_z = 1 \quad (6.15)$$

connects the Twice-Modified Kaup-Boussinesq system (6.12) and the Thrice-Modified Kaup-Boussinesq system (4.9) which has just nonlocal Hamiltonian structures, where the first of them is determined by the Poisson bracket (4.11).

Thus, the second local Hamiltonian structure (see (6.3)) of the Kaup-Boussinesq system (6.1) is the first local Hamiltonian structure (see (6.8)) of the Modified Kaup-Boussinesq system (6.7). The third local Hamiltonian structure (see (6.4)) of Kaup-Boussinesq system (6.1) is the second local Hamiltonian structure (see (6.9)) of the Modified Kaup-Boussinesq system (6.7), which is the first local Hamiltonian structure (see (6.13)) of the Twice-Modified Kaup-Boussinesq system (6.12). Moreover, the Kaup-Boussinesq system (6.1) has fourth nonlocal Hamiltonian structure (see (6.5)), which is the third nonlocal Hamiltonian structure (see (6.10)) of the Modified Kaup-Boussinesq system (6.7), also which is the second nonlocal Hamiltonian structure (see (6.14)) of the Twice-Modified Kaup-Boussinesq system (6.12), as well which is the first nonlocal Hamiltonian structure (4.10) (see (4.11)) of the Thrice-Modified Kaup-Boussinesq system (4.9) (see [9]). Thus, the Kaup-Boussinesq system (6.1) has four different Lagrangian representations

$$S_1 = \int \left[\frac{1}{2}(\psi_z^{(1)}\psi_y^{(2)} + \psi_z^{(2)}\psi_y^{(1)}) - h_4(\psi_z^{(1)}, \psi_z^{(2)}, \psi_{zz}^{(2)}) \right] dz dy, \quad (6.15)$$

$$S_2 = \int \left[-\frac{1}{2}\psi_z^{(2)}\psi_y^{(2)} + \frac{1}{8}\psi_z^{(3)}\psi_y^{(3)} - h_3(\psi_z^{(2)}, \psi_z^{(3)}, \psi_{zz}^{(3)}) \right] dz dy, \quad (6.16)$$

$$S_3 = \int \left[\frac{1}{2}(\psi_z^{(3)}\psi_y^{(4)} + \psi_z^{(4)}\psi_y^{(3)}) - h_2(\psi_z^{(3)}, \psi_z^{(4)}, \psi_{zz}^{(4)}) \right] dz dy, \quad (6.17)$$

$$S_4 = \int \left[\frac{1 - \psi_z^{(4)2}}{2\psi_z^{(5)}}\psi_y^{(5)} + \frac{1}{2}\psi_z^{(4)}\psi_y^{(4)} - h_1(\psi_z^{(4)}, \psi_z^{(5)}, \psi_{zz}^{(5)}) \right] dz dy, \quad (6.18)$$

where $\eta = \psi_z^{(1)}$, $v = \psi_z^{(2)}$, $a = \psi_z^{(3)}$, $b = \psi_z^{(4)}$, $c = \psi_z^{(5)}$ and $H_k = \int h_k dz$. Moreover, we have very interesting hierarchy:

1. The first Hamiltonian structure (see (6.2)) of the Kaup-Boussinesq system (6.1) has the Hamiltonian $H_4 = \frac{1}{2} \int [-\varepsilon^2 v_z^2 + v^2 \eta + \eta^2] dz$, the momentum $H_3 = \int v \eta dz$ and two flat coordinates (Casimirs) $H_2 = 2 \int \eta dz$, $H_1 = 2 \int v dz$.
2. The first Hamiltonian structure (see (6.8)) of the Modified Kaup-Boussinesq system (6.7) has the Hamiltonian $H_3 = \int v \eta dz = \frac{1}{4} \int [v(v^2 - a^2) - 4\varepsilon v a_z] dz$, the momentum $H_2 = 2 \int \eta dz = \frac{1}{2} \int [v^2 - a^2] dz$ and two flat coordinates $H_1 = 2 \int v dz$, $H_{-1} = \int a dz$.
3. The first Hamiltonian structure (see (6.13)) of the Twice-Modified Kaup-Boussinesq system (6.12) has the Hamiltonian $H_2 = \frac{1}{2} \int [v^2 - a^2] dz = \int [-\frac{1}{2}a^2(1 - b^2) + 2\varepsilon a b b_z + 2\varepsilon^2 b_z^2] dz$, the momentum $H_1 = 2 \int v dz = 2 \int a b dz$ and two flat coordinates $H_{-1} = \int a dz$, $H_{-2} = \int b dz$.

4. The first Hamiltonian structure (4.10) of the Thrice-Modified Kaup-Boussinesq system (4.9) has the Hamiltonian $H_1 = 2 \int abd z = 2 \int [b(1 - b^2) - 2\epsilon bc_z] dz/c$, the momentum $H_0 = \int 1 \cdot dz$ and two "geodesic" coordinates $H_{-2} = \int b dz$, $H_{-3} = \int c dz$. The generalization of the Darboux theorem on infinite-dimensional case signifies that every (local or nonlocal) Hamiltonian structure of integrable system can be reduced into canonical form "d/dx". For instance, it means that every Hamiltonian structure of the Kaup-Boussinesq system possesses a Lagrangian representation (see (6.15-18)).

Thus, here we present canonical representation for the first four Hamiltonian structures of the Kaup-Boussinesq system (6.1) (see above)

$$\begin{aligned} v_y &= \partial_z \frac{\delta H_4}{\delta \eta}, & \eta_y &= \partial_z \frac{\delta H_4}{\delta v}, \\ a_y &= -\partial_z \frac{\delta H_3}{\delta a}, & v_y &= \partial_z \frac{\delta H_3}{\delta v}, \\ b_y &= \frac{1}{2} \partial_z \frac{\delta H_2}{\delta a}, & a_y &= \frac{1}{2} \partial_z \frac{\delta H_2}{\delta b}, \\ u_t &= \frac{1}{4} \partial_x \frac{\delta H_1}{\delta u}, & w_t &= -\frac{1}{4} \partial_x \frac{\delta H_1}{\delta w}, \end{aligned}$$

which are determined by the Poisson brackets (6.2), (6.8), (6.13) and (4.16), respectively.

Conclusion.

In this article we established the Lagrangian representation for an evolutionary system, where a nonlocal Hamiltonian structure is determined by the differential-geometric Poisson bracket of the first order with metric of constant curvature. Also, we presented canonical coordinates for the first four Hamiltonian structures of the Kaup-Boussinesq system, where every of them is in canonical form "d/dx" with a Lagrangian representation. In theory of Hamiltonian structures for dispersive systems just two differential-geometric Poisson brackets of first order allow special coordinates (annihilators), where they are the exactly the same as for hydrodynamic type systems. It means that: if anyone start from Poisson bracket

$$\{u^i(x), u^k(x')\} = [a_0^{ik} \partial_x^N + a_1^{ik} \partial_x^{N-1} + \dots + a_N^{ik}] \delta(x - x'), \quad (\text{c.1})$$

where all functions (a_j^{ik}) are functions with respect to field variables u^i and their derivatives, then in some cases by special differential substitutions this expression may be transform into canonical (see (1.7) and above first three local Hamiltonian structures for Kaup-Boussinesq system, first two local Hamiltonian structures for Modified Kaup-Boussinesq system, first local Hamiltonian structure for Twice-Modified Kaup-Boussinesq system)

$$\{a^\alpha(x), a^\beta(x')\} = \bar{g}^{\alpha\beta} \delta'(x - x'). \quad (\text{c.1a})$$

It means that Poisson bracket determine Hamiltonian structure for dispersive system reducible to canonical form "d/dx". If anyone start from Poisson bracket

$$\{u^i(x), u^k(x')\} = [a_0^{ik} \partial_x^N + a_1^{ik} \partial_x^{N-1} + \dots + a_N^{ik} + \epsilon u_x^i \partial_x^{-1} u_x^k], \quad (\text{c.2})$$

where all functions (a_j^{ik}) are functions with respect to field variables u^i and their derivatives, then in some cases by special differential substitutions this expression may be transform into canonical Poisson bracket associated with metric of constant curvature (see (1.14), fourth Hamiltonian structure for Kaup-Boussinesq system, third Hamiltonian structure for Modified Kaup-Boussinesq system, second Hamiltonian structure for the Twice-Modified Kaup-Boussinesq system and first Hamiltonian structure for the Thrice-Modified Kaup-Boussinesq system)

$$\{c^\alpha(x), c^\beta(x')\} = [(\bar{g}^{\alpha\beta} - \varepsilon c^\alpha c^\beta) \partial_x - \varepsilon c^\beta c_x^\alpha + \varepsilon c_x^\alpha \partial_x^{-1} c_x^\beta] \delta(x - x'). \quad (\text{c.2a})$$

If anyone start from Poisson bracket

$$\{u^i(x), u^k(x')\} = [a_0^{ik} \partial_x^N + a_1^{ik} \partial_x^{N-1} + \dots + a_N^{ik} + \varepsilon_{\alpha\beta} w^{\alpha,i} \partial_x^{-1} w^{\beta,k},$$

where all functions ($a_j^{ik}, w^{\alpha,i}$) are functions with respect to field variables u^i and their derivatives, then **not exist any differential substitutions possessing reduction to differential-geometric nonlocal Poisson bracket of the first order** (see [10]). Thus, two Poisson brackets of arbitrary order (see (c.1 and c.2) may be reduced into differential geometric Poisson brackets of the first order arising in theory of hydrodynamic type systems (see [2], [4] and [10]). In next article, we will describe the infinite-dimensional analogue of Darboux theorem for all other nonlocal Hamiltonian structures for the Kaup-Boussinesq system and we will construct their Lagrangian representations. In this case we shall describe all types of nonlocal Hamiltonian structures where corresponding symplectic structures are local. Theory of more complicated nonlocal Hamiltonian structures were established in [10] and [11]. Our statement is that every nonlocal Hamiltonian structure determined by the differential-geometric Poisson bracket of the first order (see [10]) has a Lagrangian representation. It means that every integrable system like Kaup-Boussinesq system has an infinite set of Lagrangian representations (see for instance [12]).

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